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Schrödinger Operators in $L^2(\mathbb{R})$ with Pointwise Localized Potential

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1. INTRODUCTION

Schrödinger operators with Dirac's comb type potential have been largely studied by Albeverio *et al*. in their monograph [1] on the subject. In particular, they construct the self-adjoint realization of the formal operator $-\Delta + \Sigma k_j \delta_{aj}$ in $L^2(\mathbb{R})$, where $\{a_j\}_{j \in \mathbb{Z}}$ is a real sequence satisfying $\inf(a_{j+1} - a_j) \geq d > 0$. A study of self-adjointness under general conditions has also been done by Mikhailets in [5] (see also [6]). In these two papers, the author connects this problem to the self-adjointness of an infinite Jacobian matrix as an operator in $\ell^2(\mathbb{Z})$. He also relates the semi-boundedness of the operator to the boundedness of $\{k_j\}_{j \in \mathbb{Z}}$ and $\{(a_{j+1} - a_j)^{-1}\}_{j \in \mathbb{Z}}$. Otherwise, a self-adjoint extension in $L^2(\mathbb{R})$ of the formal operator $-\Delta + W + \Sigma k_j \delta_{aj}$ with $W \in L^1_{\text{loc}}(\mathbb{R})$ is also presented by Gesztesy and Kirsch in [3], with no conditions required on the increasing sequence $\{a_j\}_{j \in \mathbb{Z}}$. However, the weights $\{k_j\}_{j \in \mathbb{Z}}$ are determined (see Eq. (3.8) in [3]) and the authors do not present any results of lower semiboundedness in their paper.

Here, we give a rigorous rewriting of the operators $-\Delta + \Sigma k_j \delta_{aj}$ and present a direct proof of self-adjointness and lower semi-boundedness under conditions sharper than those used in [1], for the sequence $\{a_j\}_{j \in \mathbb{Z}}$, which allow $a_{j+1} - a_j$ to tend to zero. Our condition is of the type

$$\overline{\lim_{j \to \infty}} \frac{|k_{n_j}|}{a_{n_{j+1}} - a_{n_j}} < +\infty,$$

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where $\{k_{n_j}\}_{j\in\mathbb{Z}}$ is the negative subsequence of $\{k_j\}_{j\in\mathbb{Z}}$ and $\{a_{n_j}\}_{j\in\mathbb{Z}}$ is the corresponding subsequence of $\{a_j\}_{j\in\mathbb{Z}}$. Moreover, we give, explicitly, the domain of self-adjointness. We do not need any hypothesis on the positive part of $\{k_j\}_{j\in\mathbb{Z}}$ other than the boundedness of the whole sequence. The conditions required on this subsequence are, in fact, part of the domain itself. Explicitly, under the above conditions, we provide a self-adjoint realization of the operator $-\Delta + \Sigma k_j \delta_{aj}$ on

$$D = H^1(\mathbb{R}) \cap H^2(\Omega) \cap \left\{ u \Big/ \sum_{j\in\mathbb{Z}} k_{p_j} |u(k_{p_j})|^2 < \infty \right\}$$
$$\cap \left\{ u/(\forall j \in \mathbb{Z}) u'(a_j^+) - u'(a_j^-) = k_j u(a_j) \right\},$$

where $\Omega = \bigcup_{j\in\mathbb{Z}}]a_j, a_{j+1}[$.

2. SCHRÖDINGER OPERATORS WITH POINTWISE LOCALIZED POTENTIAL IN $\mathbb{R}$

Consider two real sequences $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$. Assume that $\{a_j\}_{j\in\mathbb{Z}}$ is strictly increasing and satisfies $\lim_{j\to-\infty} a_j = -\infty$, $\lim_{j\to+\infty} a_j = +\infty$, and that $\{k_j\}_{j\in\mathbb{Z}}$ is bounded. We fix the following notations:

$$(\forall j \in \mathbb{Z}) \qquad \Omega_j =]a_j, a_{j+1}[, \; \Omega = \bigcup_{j\in\mathbb{Z}} \Omega_j.$$

Define the Hermitian form B in $H^1(\mathbb{R})$ for compact supported functions in x.

$$B(u,v) = \langle u', v' \rangle_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}} k_j u(a_j) \bar{v}(a_j).$$

In order to specify the domain of definition of B, we need the following result whose proof is in [7]:

LEMMA 1. *Let the sequence* $\{a_j\}_{j\in\mathbb{Z}}$ *and* $\{k_j\}_{j\in\mathbb{Z}}$ *satisfy* $\lim_{j\to-\infty} a_j = -\infty$, $\lim_{j\to+\infty} a_j = +\infty$, $\{k_j\}_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, *and* $\overline{\lim}_{j\to\infty}(k_j/(a_{j+1} - a_j)) < +\infty$. *Then, we can find* $C > 0$ *such that, for any* $\mu > 0$, *the following holds in* $H^1(\mathbb{R})$:

$$\sum_{j\in\mathbb{Z}} k_j |u(a_j)|^2 \leq \frac{1}{\mu} \|u'\|^2_{L^2(\mathbb{R})} + C(1+\mu)\|u\|^2_{L^2(\mathbb{R})}. \tag{1}$$

Construction of the Operator. We complete the hypotheses on the sequences $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$: Denote by $\{k_{n_j}\}_{j\in\mathbb{Z}}$ the subsequence of negative terms of $\{k_j\}_{j\in\mathbb{Z}}$ and assume that $\{|k_{n_j}|\}_{j\in\mathbb{Z}}$ and $\{a_{n_j}\}_{j\in\mathbb{Z}}$ satisfy the hypotheses of Lemma 1. Hence the sum $\sum_{j\in\mathbb{Z}} k_{n_j}|u(a_{n_j})|^2$ converges. Note that we do not formulate any hypotheses other than boundedness on the positive subsequence $\{k_{p_j}\}_{j\in\mathbb{Z}}$ of $\{k_j\}_{j\in\mathbb{Z}}$. The Hermitian form B is defined on

$$\mathrm{Dom}(B) = H^1(\mathbb{R}) \cap \left\{u \Big/ \sum_{j\in\mathbb{Z}} k_{p_j}|u(a_{p_j})|^2 < \infty\right\}.$$

We have:

THEOREM 2. *Let the sequences $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$ satisfy $\lim_{j\to-\infty} a_j = -\infty$, $\lim_{j\to+\infty} a_j = +\infty$, $\{k_j\}_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, and $\overline{\lim}_{j\to\infty}(|k_{n_j}|/(a_{n_{j+1}} - a_{n_j})) < +\infty$. Define the set D by*

$$D = H^1(\mathbb{R}) \cap H^2(\Omega) \cap \left\{u \Big/ \sum_{j\in\mathbb{Z}} k_{p_j}|u(a_{p_j})|^2 < \infty\right\}$$
$$\cap \left\{u/(\forall j\in\mathbb{Z})u'\left(a_j^+\right) - u'\left(a_j^-\right) = k_j u(a_j)\right\}$$

and the operator A on D by

$$Au = -\sum_{j\in\mathbb{Z}} E_j\big(R_j(u'')\big),$$

where the operator R_j, $j\in\mathbb{Z}$, restricts a function to the open set Ω_j, while E_j, $j\in\mathbb{Z}$, extends a function of $L^2(\Omega_j)$ by 0, outside Ω_j.

Then, the operator A is lower semi-bounded, is self-adjoint with domain D, and is the unique operator satisfying

$$\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$$
$$(\forall u \in \mathrm{Dom}(A))\ (\forall v \in \mathrm{Dom}(B)) \qquad \langle Au, v\rangle_{L^2} = B(u,v).$$

Proof. Equality $\langle Au, v\rangle_{L^2} = B(u,v)$ on $D\times H^2(\mathbb{R})$, and hence the symmetry of A on D, results from an integration by parts, using a family of truncation functions in the variable x, with a uniformly bounded derivative.

Let $u_0 \in \mathrm{Dom}(A^*)$. We prove that $u_0 \in \mathrm{Dom}(A)$. First, applying

$$|\langle u_0, Av\rangle| \le C_1\|v\|_{L^2} \tag{2}$$

to test functions $v \in \mathscr{D}(\Omega) \subset D$, shows that $Au_0 \in L^2(\mathbb{R})$. Therefore,

$$Au_0 = \omega \in L^2(\mathbb{R}),$$

which is rewritten as

$$(\forall \lambda > 0) \qquad Au_0 + \lambda^2 u_0 = \omega_\lambda \in L^2(\mathbb{R}). \tag{3}$$

The parameter λ will be given a fixed value later on. For any x in Ω_j, the square integrable solutions of Eq. (3) are given by

$$\begin{aligned} u_0(x) = \frac{1}{2\lambda}\int_{a_{j+1}}^{x} e^{-\lambda(s-x)}\omega_\lambda(s)\,ds - \frac{1}{2\lambda}\int_{a_j}^{x} e^{-\lambda(x-s)}\omega_\lambda(s)\,ds \\ + \alpha_{\lambda,j}e^{-\lambda(x-a_j)} + \beta_{\lambda,j}e^{-\lambda(a_{j+1}-x)}. \end{aligned} \tag{4}$$

Denote by $u_{0,j}^c$ the integral part ("c" means "convolution") and by $u_{0,j}^h$ the rest of the right member of (4), which is the solution of the homogeneous equation associated with (3). These are the restrictions on each Ω_j of two functions denoted respectively by u_0^c and u_0^h. We are led to prove that u_0^c and u_0^h belong to $H^2(\Omega)$. Using Young's inequality $\|f * g\|_{L^2} \le \|f\|_{L^1}\|g\|_{L^2}$ we prove that for any j in $\mathbb{Z}$,

$$\|u_{0,j}^c\|_{L^2(\Omega_j)} \le \frac{1}{\lambda^2}\|Rj(\omega)\|_{L^2(\Omega_j)},$$

$$\left\|\frac{du_{0,j}^c}{dx}\right\|_{L^2(\Omega_j)} \le \frac{1}{\lambda}\|Rj(\omega)\|_{L^2(\Omega_j)}, \tag{5}$$

$$\left\|\frac{d^2u_{0,j}^c}{dx^2}\right\|_{L^2(\Omega_j)} \le 2\|Rj(\omega)\|_{L^2(\Omega_j)}. \tag{6}$$

On the other hand, we can also write

$$\|u_{0,j}^h\|_{L^2(\Omega_j)} \le \frac{\left(\alpha_{\lambda,j}^2 + \beta_{\lambda,j}^2\right)^{1/2}}{\lambda^{1/2}},$$

$$\left\|\frac{du_{0,j}^h}{dx}\right\|_{L^2(\Omega_j)} \le \left[\lambda\left(\alpha_{\lambda,j}^2 + \beta_{\lambda,j}^2\right)\right]^{1/2}, \tag{7}$$

$$\left\|\frac{d^2u_{0,j}^h}{dx^2}\right\|_{L^2(\Omega_j)} \le \left[\lambda^3\left(\alpha_{\lambda,j}^2 + \beta_{\lambda,j}^2\right)\right]^{1/2}. \tag{8}$$

Therefore, we have $u_0^c(x) \in H^2(\Omega)$ with $\|u_0^c(x)\|_{H^2(\Omega)} \leq C_2 \|\omega\|_{L^2(\mathbb{R})}$ and for any φ in $\mathscr{D}(\mathbb{R})$ $(\varphi(x) \cdot u_0^h(x))_{|\Omega} \in H^2(\Omega)$. However, as we do not know whether $\sum_{j \in \mathbb{Z}} (a_{\lambda, j}^2 + \beta_{\lambda, j}^2)$ converges or not, it is not possible to establish any global property for $u_0^h(x)$ (i.e., a property on the whole $\mathbb{R}$).

The following boundary conditions hold for any j in $\mathbb{Z}$:

$$u_0(a_j^+) = u_0(a_j^-); \qquad \frac{\partial u_0}{\partial x}(a_j^+) - \frac{\partial u_0}{\partial x}(a_j^-) = k_j u_0(a_j).$$

Let $\varphi_j \in D$ be such that $\varphi_j(a_j) = 0$ and $\operatorname{supp}(\varphi_j) \subset]a_{j-1}, a_{j+1}[$. Integrating $\langle u_0, A\varphi_j \rangle_{L^2}$ by parts in (2), we can write

$$|\overline{\varphi}_j'(a_j)\big(u_0(a_j^+) - u_0(a_j^-)\big)| \leq C_{3,j} \|\varphi_j\|_{L^2}, \tag{9}$$

with

$$C_{3,j} = \|u_0''|_{]a_{j-1}, a_{j+1}[}\|_{L^2(]a_{j-1}, a_{j+1}[)} + C_1 < +\infty.$$

Let $\varphi \in \mathscr{D}(\mathbb{R})$ be such that $\varphi(0) = 0$, $\varphi'(0) = 1$. For any $j \in \mathbb{Z}$, define $\varphi_{j,\varepsilon}$ in $\mathbb{R}$ by

$$(\forall \varepsilon \in]0, \delta_j[)\ (\forall x \in \mathbb{R}) \qquad \varphi_{j,\varepsilon}(x) = \varphi\left(\frac{x - a_j}{\varepsilon}\right).$$

Taking the constant $\delta_j < 1$ ensures that $\operatorname{supp}(\varphi_{j,\varepsilon}) \subset]a_{j-1}, a_{j+1}[$, so that $\varphi_{j,\varepsilon}$ belongs to D. Using the inequality (9) with the family $\varphi_{j,\varepsilon}$, we obtain

$$|u_0(a_j^+) - u_0(a_j^-)| \leq \varepsilon^{3/2} C_{3,j} \|\varphi\|_{L^2}.$$

Finally, letting ε tend to 0 proves that u_0 is continuous at the point a_j.

We shall now prove the boundary conditions

$$(\forall j \in \mathbb{Z}) \qquad u_0'(a_j^+) - u_0'(a_j^-) = k_j u_0(a_j).$$

Let $j \in \mathbb{Z}$ and $\psi_j \in D$ be such that $\psi_j(a_j) \neq 0$ and $\operatorname{supp}(\psi_j) \subset]a_{j-1}, a_{j+1}[$. Proceeding as for the inequality (9) and using the continuity of u_0, we obtain

$$|\overline{\psi}_j(a_j) \cdot \big(u_0'(a_j^+) - u_0'(a_j^-) - k_j u_0(a_j)\big)| \leq C_{3,j} \|\psi_j\|_{L^2}. \tag{10}$$

Let $\varphi \in \mathscr{D}(\mathbb{R})$, $\varphi(x) = 1$, in a neighbourhood of the origin. Defining the family $\psi_{j,\varepsilon}$ in $\mathbb{R}$ by

$$\psi_{j,\varepsilon}(x) = \varphi\left(\frac{x - a_j}{\varepsilon}\right)\big(1 + k_j \cdot (x - a_j) \cdot H(x - a_j)\big),$$

the conditions on the parameter are the same as before to ensure that each $\psi_{j,\varepsilon}$ belongs to D. Applying inequality (10) to this family of functions, we obtain

$$|u_0'(a_j^+) - u_0'(a_j^-) - k_j u_0(a_j)| \le C_{3,j} \cdot K_j \cdot \varepsilon^{1/2} \|\varphi\|_{L^2(\mathbb{R})},$$

where, for any j in $\mathbb{Z}$,

$$K_j = 1 + |k_j|(a_{j+1} - a_j).$$

As ε tends to 0, we reach

$$u_0'(a_j^+) - u_0'(a_j^-) = k_j u_0(a_j).$$

We now show that $\alpha_{\lambda,j}$ and $\beta_{\lambda,j}$ satisfy

$$\sum_{j\in\mathbb{Z}} \alpha_{\lambda,j}^2 + \beta_{\lambda,j}^2 < \infty.$$

Replacing u_0 by its restrictions in the preceding boundary conditions, we obtain

$$\left.\begin{aligned} u_{0,j}(a_j^+) &= u_{0,j-1}(a_j^-) \\ \frac{du_{0,j}}{dx}(a_j^+) - \frac{du_{0,j-1}}{dx}(a_j^-) &= k_j P_j u_{0,j}(a_j). \end{aligned}\right\} \tag{11}$$

Denote

$$e_{\lambda,j} = e^{-\lambda(a_{j+1}-a_j)}$$

$$A_{\lambda,j} = \int_{a_{j-1}}^{a_j} e^{-\lambda(a_j - s)} \omega_{j-1}(s)\, ds$$

$$B_{\lambda,j} = \int_{a_j}^{a_{j+1}} e^{-\lambda(s-a_j)} \omega_j(s)\, ds.$$

The equations (11) can be rewritten as

$$M_\lambda \cdot \begin{pmatrix} \cdots \\ \beta_{\lambda,j} \\ \alpha_{\lambda,j} \\ \cdots \end{pmatrix} = \frac{1}{2\lambda} \begin{pmatrix} \cdots \\ (k_j/2\lambda) B_{\lambda,j} - A_{\lambda,j} \\ (k_{j+1}/2\lambda) A_{\lambda,j+1} - B_{\lambda,j+1} \\ \cdots \end{pmatrix},$$

where M_λ is the following matrix:

$$M_\lambda = \begin{pmatrix} & \cdots & & & \mathbf{O} \\ & \left(1+\frac{k_j}{2\lambda}\right) & \frac{k_j}{2\lambda}e_{\lambda,j-1} & -e_{\lambda,j} & \\ & & -e_{\lambda,j-1} & \frac{k_j}{2\lambda}e_{\lambda,j} & \left(1+\frac{k_j}{2\lambda}\right) \\ & \mathbf{O} & & & \cdots \end{pmatrix}.$$

We can write $M_\lambda = R_\lambda + \frac{1}{2\lambda}S_\lambda$, with R_λ and S_λ defined by

$$R_\lambda = \begin{pmatrix} \cdots & & & & \\ \mathbf{O} & \begin{bmatrix} 0 & -e_{\lambda,j-1} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ -e_{\lambda,j+1} & 0 \end{bmatrix} & \mathbf{O} \\ & & & & \cdots \end{pmatrix}$$

$$S_\lambda = \begin{pmatrix} \cdots & & \\ & \begin{bmatrix} k_j e_{\lambda,j} & k_j \\ k_{j+1} & k_{j+1}e_{\lambda,j} \end{bmatrix} & \\ & & \cdots \end{pmatrix}.$$

The matrix R_λ is explicitly invertible:

$$R_\lambda^{-1} = \begin{pmatrix} \cdots & & & & \\ \mathbf{O} & \begin{bmatrix} 0 & 0 \\ e_{\lambda,j-1} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & e_{\lambda,j+1} \\ 0 & 0 \end{bmatrix} & \mathbf{O} \\ & & & & \cdots \end{pmatrix}.$$

Recall that if we denote by $\| \ \|_{\mathscr{L}^p}$ the subordinate matrix norms to the vector norms $\| \ \|_p$, we have

$$\|A\|_{\mathscr{L}^\infty} = \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a_{ij}|, \qquad \|A\|_{\mathscr{L}^1} = \sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} |a_{ij}|, \qquad \text{and}$$

$$\|A\|_{\mathscr{L}^2} \le \sqrt{\|A\|_{\mathscr{L}^1}\|A\|_{\mathscr{L}^\infty}}\,.$$

We can easily prove that $\|R_\lambda\|_{\mathscr{L}^2} \le 2$, $\|R_\lambda^{-1}\|_{\mathscr{L}^2} \le 2$, and $\|S_\lambda\|_{\mathscr{L}^2} \le 2\|k_j\|_{\ell^\infty}$. If $\lambda > 2\|k_j\|_{\ell^\infty}$, then M_λ is invertible and we have

$$\|M_\lambda^{-1}\|_{\mathscr{L}^2} \le \|R_\lambda^{-1}\|_{\mathscr{L}^2} \frac{1}{1 - \frac{1}{2\lambda}\|R_\lambda^{-1}\|_{\mathscr{L}^2}\|S_\lambda\|_{\mathscr{L}^2}} \le \frac{2\lambda}{\lambda - 2\|k_j\|_{\ell^\infty}}.$$

Denoting the right member of this inequality by $c(\lambda)$ and fixing a suitable value $\lambda_0 > 2k\|k_j\|_{\ell^\infty}$ for λ, we obtain

$$\left\| \begin{pmatrix} \cdots \\ \beta_{\lambda_0, j} \\ \alpha_{\lambda_0, j} \\ \cdots \end{pmatrix} \right\|_2 \leq \left(\frac{c(\lambda_0)}{2\lambda_0} \right) \left\| \begin{pmatrix} \cdots \\ (k_j/2\lambda_0) B_{\lambda_0, j} - A_{\lambda_0, j} \\ (k_{j+1}/2\lambda_0) A_{\lambda_0, j+1} - B_{\lambda_0, j+1} \\ \cdots \end{pmatrix} \right\|_2 . \quad (12)$$

For any j in $\mathbb{Z}$, simple computations yield

$$A_{\lambda_0, j} \leq \frac{1}{\sqrt{2\lambda_0}} \left(\int_{a_{j-1}}^{a_j} \omega^2_{\lambda_0, j-1}(s)\, ds \right)^{1/2},$$

and the corresponding inequality for $B_{\lambda_0, j}$. From (12), we deduce that

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \alpha^2_{\lambda_0, j} + \beta^2_{\lambda_0, j} &\leq \frac{c^2(\lambda_0)}{(2\lambda_0)^3} \left(\sum_{j\in\mathbb{Z}} \left(\frac{|k_j|^2}{(2\lambda_0)^2} + 1 \right) \left(\int_{a_{j-1}}^{a_j} \omega^2_{\lambda_0, j-1}(s)\, ds \right) \right) \\ &\leq \frac{c^2(\lambda_0)}{(2\lambda_0)^3} \left(\frac{\|k_j\|^2_{\ell^\infty}}{(2\lambda_0)^2} + 1 \right) \|\omega_{\lambda_0}\|^2_{L^2(\mathbb{R})}, \end{aligned} \quad (13)$$

which gives the desired result.

The function u_0 satisfies $d^2u_0/dx^2|\Omega \in L^2(\Omega)$, $du_0/dx|\Omega \in L^2(\mathbb{R})$.

The first property is a consequence of inequalities (5), (7), and (13). For the second one, we first use (6), (8), and (13) in order to obtain

$$\frac{du_0}{dx}|\Omega \in L^2(\Omega).$$

Then for any φ in $\mathscr{D}(\mathbb{R}^n)$, we have

$$\left| \left\langle \frac{du_0}{dx}, \varphi \right\rangle_{\mathscr{D}'\mathscr{D}} \right| = - \left\langle u_0, \frac{d\varphi}{dx} \right\rangle_{L^2}.$$

Integrating $\langle u_0, \frac{d\varphi}{dx} \rangle_{L^2}$ by parts on each I_j and using the continuity of u_0 at the a_j's, we obtain

$$\begin{aligned} \left| \left\langle \frac{du_0}{dx}, \varphi \right\rangle_{\mathscr{D}'\mathscr{D}} \right| &= \left\langle \frac{du_0}{dx}|\Omega, \varphi|\Omega \right\rangle_{L^2} \\ &\leq \|u_0|\Omega\|_{H^1(\Omega)} \|\varphi\|_{L^2}. \end{aligned}$$

The conclusion follows. Hence, $u_0 \in H^1(\mathbb{R})$.

Let us prove that the sum $\sum_{j\in\mathbb{Z}} k_{p_j}|u_0(a_{p_j})|^2$ is finite.

Let $(\forall \ell \in \mathbb{N})$ $\varphi_\ell \in \mathscr{D}(]-\ell-1, \ell+1[)$ be such that $0 \le \varphi_\ell(x) \le 1$, in $\mathrm{supp}(\varphi_\ell)$, $\varphi_{\ell\,]-\ell,\ell[} = 1$ and φ_ℓ' being uniformly bounded in ℓ. We note $u_{0,\ell}(x) = u_0(x)\varphi_\ell(x)$. An integration by parts yields, for any ℓ in $\mathbb{N}$,

$$\langle Au_0, u_{0,\ell}\rangle_{L^2(\mathbb{R})} = \left\langle \frac{du_0}{dx}, \frac{du_{0,\ell}}{dx}\right\rangle_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}} k_j u_0(a_j)\overline{u_{0,\ell}}(a_j).$$

Lemma 1 gives

$$\sum_{j\in\mathbb{Z}} |k_{n_j}|\,|u_0(a_{n_j})|^2 \le C_4\|u_0\|^2_{H^1(\mathbb{R})}.$$

Therefore,

$$\sum_{j\in\mathbb{Z}} k_{p_j}\varphi_\ell(a_{p_j})|u_0(a_{p_j})|^2\Big) \le C\|u_0\|^2_{H^1(\mathbb{R})} + \langle Au_0, u_{0,\ell}\rangle_{L^2(\mathbb{R})} - \left\langle \frac{du_0}{dx}, \frac{du_{0,\ell}}{dx}\right\rangle_{L^2(\mathbb{R})}.$$

We conclude the proof using the monotone convergence theorem for the discrete sum and Lebesgue's theorem for the terms in between brackets.

Finally, we have shown that $u_0 \in D$. Thus $(A, D, L^2(\mathbb{R}))$ is a self-adjoint operator. Its uniqueness follows from a classical result about self-adjoint operators associated with quadratic forms (see [8, Theorem VIII.15]). From inequality (1), we can find a constant $C > 0$ such that, in $\mathrm{Dom}(B)$, the following holds:

$$\sum_{j\in\mathbb{Z}} |k_{n_j}|\,|u(a_{n_j})|^2 \le \|\nabla u\|^2_{L^2} + 2C\|u\|^2_{L^2}. \tag{14}$$

Hence B is lower semi-bounded by $-2C$ in $\mathrm{Dom}(B)$. There remains to prove that $(\mathrm{Dom}(B), \|\cdot\|_B)$ is closed (with $\|\cdot\|_B = (B(\cdot,\cdot) + (2C+1)\|\cdot\|^2_{L^2})^{1/2}$).

The norms $\|\cdot\|_B$ and $\|\cdot\|_{H^1(\mathbb{R}^n)}$ are not equivalent, as no hypothesis was made on the positive subsequence $\{k_{p_j}\}_{j\in\mathbb{Z}}$ of $\{k_j\}_{j\in\mathbb{Z}}$, with respect to $k_j/(a_{j+1}-a_j)$. However, using inequality (1) with $\mu = 1 + \frac{1}{2C}$ we prove that

$$\|\cdot\|^2_B \ge \min\left(\frac{1}{2}, \frac{4\pi^2}{2C+1}\right)\|\cdot\|^2_{H^1(\mathbb{R})}.$$

Let $\{u_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence in Dom(B) for $\|\ \|_B$. It converges to a function u in $(H^1(\mathbb{R}), \|\ \|_{H^1})$ and we have, for any j in $\mathbb{Z}$,

$$\lim_{m\to+\infty} |u_m(a_j) - u(a_j)| = 0. \tag{15}$$

Besides, using (14), we have

$$\sum_{j\in\mathbb{Z}} |k_{n_j}|\,|u(a_{n_j})|^2 < +\infty.$$

On the other hand, the sequence (indexed by m) $\{k_{pj}|u_m(a_{p_j})|^2\}_{j\in\mathbb{Z}}$ is a Cauchy sequence in $\ell^1(\mathbb{Z})$, since

$$\sum_{j\in\mathbb{Z}} k_{p_j}|u_m(a_{p_j})|^2 \leq \|u_m\|_B.$$

From Eq. (15) we deduce that the limit sequence is, in fact, equal to $\{k_{p_j}|u(a_{p_j})|^2\}_{j\in\mathbb{Z}}$ and that the latter belongs to $\ell^1(\mathbb{Z})$. Consequently, a Cauchy sequence in $(\mathrm{Dom}(B), \|\ \|_B)$ is convergent in this set, which concludes the proof.

Remarks. This method applies in the same way to a finite sequence of points $\{a_j\}_{j\in\mathbb{Z}}$. In this case the only requirement is that the sequence of weights $\{k_j\}_{j\in\mathbb{Z}}$ should be bounded.

A similar result can be constructed in $\mathbb{R}^q$ with a potential localized on parallel hyperplanes. Its proof follows the same principle modulo a few technical arrangements:

The adequate operator is $A_q u = -\sum_{j\in\mathbb{Z}} Ej(Rj(\Delta u))$, where Rj and Ej relatively are the restriction and extension operators to the open set $\Omega_{q,j} = \mathbb{R}^{q-1} \times\,]a_j, a_{j+1}[$. Denote $\Omega_q = \bigcup_{j\in\mathbb{Z}} \Omega_{q,j}$. The operator A_q is to be studied in the set

$$D = H^1(\mathbb{R}^q) \cap H^2(\Omega_q) \cap \left\{ u \Big/ \sum_{j\in\mathbb{Z}} k_{p_j}\|\nu_{p_j}u\|^2_{L^2(\mathbb{R}^{q-1})} < \infty \right\}$$
$$\cap \left\{ u \Big/ (\forall j\in\mathbb{Z})\, \frac{\partial u}{\partial N_j^+} + \frac{\partial u}{\partial N_j^-} = -k_j \nu_j u \right\}$$

with ν_j the trace operator on the hyperplane $\{x_q = a_j\}$ and the normal derivatives

$$\frac{\partial u}{\partial N_j^+} = -\nu_j\left(\frac{\partial u}{\partial x_q}|\Omega_j\right), \qquad \frac{\partial u}{\partial N_j^-} = \nu_j\left(\frac{\partial u}{\partial x_q}|\Omega_{j-1}\right).$$

A partial Fourier transformation in the first $q-1$ variables, denoted by "$\wedge$", is applied to the equation $A_q u_0 = \omega \in L^2(\mathbb{R}^q)$ in order to obtain the equivalent of Eq. (3) in $\mathbb{R}^q$:

$$A\hat{u}_0 + (2\pi)^2\{\xi'\}^2\hat{u}_0 = \hat{\omega}_\lambda \in L^2(\mathbb{R}^q),$$

with

$$(\forall u \in D) \quad A = -\sum_{j\in\mathbb{Z}} Ej\left(Rj\left(\frac{\partial^2 u}{\partial x_q^2}\right)\right) \quad \text{and} \quad \{\xi'\} = \left(\lambda^2 + |\xi|^2\right)^{1/2}.$$

The solution of this equation is as follows:

$$\begin{aligned}\hat{u}_0(\xi', x_q) = {} & \frac{1}{4\pi\langle\xi'\rangle}\int_{a_{j+1}}^{x_q} e^{-2\pi\langle\xi'\rangle(s-xq)}\hat{\omega}(\xi', s)\,ds \\ & - \frac{1}{4\pi\langle\xi'\rangle}\int_{a_j}^{x_q} e^{-2\pi\langle\xi'\rangle(xq-s)}\hat{\omega}(\xi', s)\,ds \\ & + \alpha_j(\xi')e^{-2\pi\langle\xi'\rangle(xq-aj)} + \beta_j(\xi')e^{-2\pi\langle\xi'\rangle(aj+1-xq)}.\end{aligned}$$

Fix $\mathscr{O}$, a bounded open set in $\mathbb{R}^{q-1}$, and ψ, a function in $\mathscr{D}(\mathscr{O})$. The function of one variable $\langle\hat{u}_0, \psi\rangle_{L^2(\mathbb{R}^{q-1})}$ has the same regularity and satisfies the same trace equalities as in the monodimensional case. Hence, we prove local regularity for u_0.

Global regularity is obtained by showing that $\sum_{j\in\mathbb{Z}}\alpha_j(\xi')^2 + \beta_j(\xi')^2$ satisfies some integration properties. The method is the same as in the monodimensional case (inversion of an infinite system in $\ell^2(\mathbb{Z})$) and leads to the inequality

$$\sum_{j\in\mathbb{Z}}\alpha_j(\xi')^2 + \beta_j(\xi')^2 \le \frac{c^2(\lambda)}{(4\pi\{\xi'\})^3}\left(\frac{\|k_j\|_{\ell^\infty}^2}{(4\pi)^2} + 1\right)\|\omega_\lambda(\xi', \cdot)\|_{L^2(\mathbb{R})}^2,$$

with

$$c(\lambda) = \frac{2\pi\lambda}{\pi\lambda - \|k_j\|_{\ell^\infty}}.$$

There remains to prove the uniqueness of A_q with Theorem VIII.15 of [8], using a q-dimensional version of inequality (1) in order to conclude.

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